MATC63 2022 Assignment 1 – Solutions

(1) Let α be a circle of radius r, say
$$\alpha(t) = (r\cos(t), r\sin(t), 0).$$

(a) Find an arc length parametrization of α.

Solution:
$$\alpha'(t) = (-r\sin(t), r\cos(t), 0)$$
$$|\alpha'(t)| = r$$
$$s(t) = \int_0^t |\alpha'(u)|du = rt$$
$$t = s/r$$

So the arc length parametrization is
$$A(s) = (r\cos(s/r), r\sin(s/r), 0).$$

(b) Prove that the curvature of α is $1/r$.

Using the arclength parametrization,
$$T = A' = (-r/r)\sin(s/r), (r/r)\cos(s/r), 0) = (-\sin(s/r), \cos(s/r), 0)$$
$$dT/ds = \big(-(1/r)\cos(s/r), -(1/r)\sin(s/r), 0\big).$$
$$T' = \kappa N$$

by Frenet 1 so
$$\kappa = |dT/ds| = 1/r.$$

(2) Find the Frenet apparatus of the curve
$$\alpha(t) = (t, 2\sqrt{t^2-1}, 1).$$

(This curve is part of the hyperbola
$$\{(X,Y,Z) : X^2 - Y^2 = c, Z = d\}$$
where c and d are real constants.)

Compute only T, B and τ. You do not need to explicitly compute the expressions for N and κ. You will need some properties of N to compute B.

Note that this curve is not arclength parametrized, so you need to use the Frenet equations for non arclength parametrized curves.

Solution:
$$\alpha' = vT$$

where $v(t) = ds/dt$. Here

$$\alpha' = (1, t(t^2-1)^{-1/2}, 0)$$

So

$$v^2 = |\alpha'|^2 = \frac{2t^2-1}{t^2-1}.$$

Note that

$$T = \alpha'/|\alpha'| = \Big(\frac{t^2-1}{2t^2-1}\Big)^{1/2}((1, \frac{t}{\sqrt{t^2-1}}, 0).$$

$$\alpha'' = (0, (t^2-1)^{-1/2} - \frac{1}{2}t(t^2-1)^{-3/2}, 0)$$

Note that α' and α'' have their third coordinate equal to 0. Hence the third coordinate of N is also 0.

$$B = T \times N$$

Since T and N both have their third coordinate equal to 0, $B = T \times N = (0,0,1)$ (or $(0,0,-1)$). B is constant (since it is a continuous function on a connected set) so $dB/dt = 0$ so $\tau = 0$.

This is consistent with the fact that this curve is a plane curve.

$$v\kappa = |dT/dt|$$
$$dB/dt = -v\tau N$$

so

$$v\tau = - < N, dB/dt >$$

(3) Shifrin chap. 1.1 5(b):

The catenary is

$$\alpha(t) = (t, \cosh(t)).$$

Reparametrize it by arc length.

You may use the result of 5.5 (a) that the arclength of $\alpha(t)$ for $t \in [0, b]$ is $\sinh(b)$.

Solution:

$$s(b) = \sinh(b)$$
$$\cosh^2(b) - \sinh^2(b) = 1$$

So

$$\cosh(t) = \sqrt{1 + \sinh^2(b))} = \sqrt{1+s^2}$$

Also

$$t = \cosh^{-1}\big(\cosh(t)\big) = \cosh^{-1}(\sqrt{1+s^2}$$

(note that cosh is bijective)

Because $e^t - e^{-t} = 2s$, we have

$$u^2 - 2su - 1 = 0$$

(letting $u = e^t$) so we can solve for u using the quadratic formula:

$$u = s + \sqrt{s^2 + 1}$$

(there is a second solution, which is disregarded because it is negative but u must be positive)

Hence we get

$$t = \ln(s + \sqrt{s^2 + 1})$$

Hence our parametrization of $\alpha(t) = (t, \cosh t)$ is

$$\left(\ln(s + \sqrt{s^2 + 1}), \sqrt{1 + s^2}\right).$$

(4) Curvature of an arclength parametrized curve – do Carmo section 1.5 no. 1 (a), (b) p. 22

Given the parametrized curve (helix)

$$\alpha(s) = (a \cos s/c, a \sin s/c, bs/c)$$

where $c^2 = a^2 + b^2$,

(a) Show that α is parametrized by arc length

Proof: The arc length is

$$d\alpha/ds = (-a/c \sin(s/c), a/c \cos(s/c), b/c)$$

$$|d\alpha/ds|^2 = a^2 + b^2)/c^2 = 1$$

It follows that the curve is parametrized by arclength.

(5) Frenet apparatus for a curve which is not necessarily arclength parametrized – Shifrin Section 1.2 3(e)

Compute the Frenet apparatus for

$$\alpha(t) = (\cosh(t), \sinh(t), t)$$

Solution:

Define $c = \cosh(t)$, $s = \sinh(t)$.

So $\alpha(t) = (c, s, t)$.

$$d\alpha/dt = (s, c, 1)$$

$$d^2\alpha/dt^2 = (c, s, 0)$$

$$T = (1/v)\frac{d\alpha}{dt} = v^{-1}(s, c, 1)$$

(from now on all derivatives are w.r.t t, not s, and ′ means d/dt)

$$dT/dt = -(1/v^2)v'\alpha' + (1/v)\alpha''$$

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = v\frac{dT}{ds}$$

Note that

$$v^2 = 1 + s^2 + c^2$$

so

$$2vv' = 2ss' + 2cc' = 2sc + 2cs = 4sc$$

By Frenet 1,

$$\frac{dT}{ds} = \kappa N$$

so

$$\kappa = |dT/ds| = |1/v dT/dt|.$$

In other words,

$$\kappa = (1/|v|)|T'|,$$
$$T = (1/v)\alpha',$$
$$T' = (-v'/v^2)\alpha' + (1/v)\alpha''.$$

Step 1:

$$v = ds/dt = |d\alpha/dt| = \sqrt{1 + s^2 + c^2}$$

$$v^2\kappa^2 = (T')^2 = (\frac{(v')^2}{v^4}|\alpha'|^2 + 1/v^2|\alpha''|^2 - 2\frac{v'}{v^3}(\alpha', \alpha'').$$

$$\alpha' = (s, c, 1)$$
$$\alpha'' = (c, s, 0)$$
$$|\alpha''|^2 = c^2 + s^2$$
$$|\alpha'|^2 = 1 + s^2 + c^2$$
$$(\alpha', \alpha'') = 2sc$$

$$N = T'/|T'|$$

$$= T'/v\kappa$$

$$B = T \times N$$

(cross product)

$$= (1/v^2)(\alpha' \times \Big((-v'/v)\alpha' + \alpha''\Big)$$

$$= (1/v^2)(\alpha' \times \alpha'')$$
$$= (1/v^2)(-s, c, -1)$$
$$dB/dt = v\tau N.$$

$$dB/dt = (-2v'/v^3)(-s, c, -1) + (1/v^2)(-c, s, 0)$$

To find τ we take the dot product of dB/dt with N. This is

$$B' \cdot N = (-2v'/v^3)(1/v\kappa)(-s, c, -1) \cdot T' + (-1/v^2)(1/v\kappa)(-c, s, 0) \cdot T'.$$

Now

$$T' = (-v'/v^2)\alpha' + (1/v)\alpha''$$

and

$$\alpha'' = (c, s, 0)$$

while

$$\alpha' = (s, c, 1).$$

So

$$B' \cdot N = (-2v'/v^3)(1/v\kappa)(-v'/v^2)(-s, c, -1) \cdot \alpha' + (-2v'/v^3)(1/v\kappa)(1/v)(-s, c, -1) \cdot \alpha''$$
$$+ (-1/v^2)(1/v\kappa)(-v'/v^2)(-c, s, 0) \cdot \alpha' + (-1/v^2)(1/v\kappa)(1/v)(-c, s, 0) \cdot \alpha''$$

Now use that

$$(-s, c, -1) \cdot \alpha' = 0$$
$$(-c, s, 0) \cdot \alpha' = 0$$
$$(-s, c, -1) \cdot \alpha'' = 0$$
$$(-c, s, 0) \cdot \alpha'' = -1$$

so we get

$$-\tau = B' \cdot N = (-1/v^2)(1/v\kappa)(1/v)(-1) = (1/v^4\kappa)$$

where v and κ were computed above.

(6) (cf. Shifrin 1.2 no. 18)

Suppose α is a helix with axis in direction A. Let β be the curve obtained by projecting α onto a plane orthogonal to A.

(a) Prove that the principal normals of α and β are parallel at corresponding points.

Proof: WLOG take A to be the z axis so β is the curve obtained by projecting α on the (x, y) plane (in other words if $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ then $\beta = (\alpha_1, \alpha_2)$. The principal normal is $T'/|T'|$ where $T = \alpha'$. Here

$$\alpha = (a\cos(s/c), a\sin(s/c), bs/c)$$

For α, $\alpha' = T$ is given in Shifrin §1.2, Example 1:

$$N = (-\cos(s/c), -\sin(s/c), 0)$$

$$\beta = (a\cos(s/c), a\sin(s/c))$$
$$T(\beta) = \beta'/|\beta'| = 1/c(-\sin(s/c), \cos(s/c))$$
$$T'(\beta) = \frac{1}{c^2}(-\cos(s/c), -\sin(s/c))$$

As claimed, $T'(\beta)$ is parallel to $N(\alpha)$.

(b) Calculate the curvature of β in terms of the curvature of α.

Solution: For α as shown in Shifrin §1.2 Example 1, $\kappa = a/c^2$.

To find the curvature of β, we rewrite $T(\beta)$ as having unit length:

$$T(\beta) = (-\sin(s/c), \cos(s/c))$$

So the curvature is the length of T'. But $T' = \frac{1}{c}(-\cos(s/c), -\sin(s/c), 0)$ So the curvature of β is $1/c$.

(7) Isoperimetric inequality: – do Carmo section 1-7 no. 1

(a) Can there be a rectangle of area 9 and perimeter 4?

Solution: $L^2 \geq 4\pi A$ (isoperimetric inequality) $16 \geq 4\pi(9)$ $4 \geq 9\pi$ This is not possible.

(b) What is the largest area that can be enclosed in a rectangle of perimeter $2P$, where $P > 0$ is a real number? What are the side lengths of this rectangle?

Solution: The side lengths of a rectangle are x and y with $2x + 2y = 2P$, so $y = P - x$. The area is $x(P-x) = Px - x^2 = A(x)$. The maximum area is achieved when $d/dx(A(x)) = P - 2x = 0$ so $x = P/2$ and $y = P/2$. So the rectangle of maximal area is a square.

(8) (do Carmo section 1.5 question 13) Let α be a curve parametrized by arclength with $\kappa, \tau \neq 0$. Suppose α lies on the surface of a sphere centered at the origin.

Let $\rho = 1/\kappa$ and $f = 1/\tau$. Differentiate $|\alpha(s)|$ to order 3 w.r.t. s and use the fact that (T, N, B) is a basis for $\mathbf{R^3}$ to show

$$\alpha(s) = -\rho N + \rho' f B.$$

This is equivalent to (a)

$$< \alpha, N >= -1/\kappa$$

and (b)

$$< \alpha, B >= (1/\kappa)'(1/\tau).$$

Solution:

Suppose $< \alpha, \alpha >= 1$. We assume the curve is arclength parametrized.

Differentiating w.r.t. s we also get $< \alpha, \alpha' > 0 = 0$. Since $\alpha' = T$ this means $\alpha = uN + vB$ for some functions $u(s)$ and $v(s)$.

It follows that

$$0 = d/ds < \alpha, \alpha' > = < \alpha, \alpha'' > + < \alpha', \alpha' > .$$

Since $\alpha' = T$ it follows that

$$< \alpha, \alpha'' > = -1.$$

But $\alpha' = T$ and $\alpha'' = T' = \kappa N$ (by Frenet 1) so

$$< \alpha, \alpha'' > = \kappa < \alpha, N > = -1,$$

as claimed.

Now it follows that

$$< \alpha, \alpha''' > = 0.$$

This means

$$< \alpha, d/ds(\kappa N) > = 0.$$

$$< \alpha, \kappa' N + \kappa N' > = 0$$

$$< \alpha, \kappa' N + \kappa(-\kappa T + \tau B) > = 0$$

Since $< \alpha, N > = -1/\kappa$,

$$\kappa'(-1/\kappa) + \kappa\tau < \alpha, B > = 0.$$

The result for (b) follows.